ON PERIODIC SOLUTIONS OF SECOND ORDER DYNAMIC SYSTEMS NEAR PIECE-WISE LINEAR ONES

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Following Neimark [1-3], the author obtains necessary and sufficient conditions for stability of various possible types of periodic motions for a particular kind of second order dynamic systems near piece-wise linear ones that are of special interest in applications.

1. Nonautonomous systems near piece-wise linear ones. We shall consider the system

$$dx / dt = y, \qquad dy / dt = -\psi(x) + \mu f(x, y, t) \tag{1.1}$$

Let

$$\psi(x) \equiv \alpha_i x + \beta_i \quad \text{if } x_{i-1} < x < x_i \qquad (i = \ldots -1, 0, 1, 2, \ldots)$$

$$f(x, y, t) \equiv f_i^{(1)}(x, y, t) \quad \text{if } x_{i-1} < x < x_i,\ y > 0$$

$$f(x, y, t) \equiv f_i^{(2)}(x, y, t) \quad \text{if } x_{i-1} < x < x_i,\ y < 0$$

Here the $f_i^{(j)}(x, y, t)$ $(j = 1, 2)$ are analytic functions of period 2π in t, and μ is a small positive parameter. We assume that the system (1.1), with $\mu = 0$, has at the origin $(x = y = 0)$ of the coordinate system an equilibrium of the type of a center, or a "sewn center."

If $\mu \neq 0$, the phase coordinates of the system (1.1) will be x, y and t. The points with the same x and y coordinates on the planes $t = t_0$ and $t = t_0 + 2\pi n$ will be considered to be identical.

We shall denote by $S_k^{(1)}$ the half-planes $x = x_k$ when $y > 0$, and by $S_k^{(2)}$ the half-planes when $y < 0$ $(k = \ldots -1, 0, 1 \ldots)$. Let us consider the trajectories of the system (1.1) with $\mu = 0$ and with $\mu \neq 0$, which satisfy the same initial conditions

$$x = x_0, \quad y = y_0 > 0 \quad \text{if } t = t_0 \tag{1.2}$$

We introduce a new time $t = \tau + t_0$ and consider the point transformation of mapping the half-plane $S_0^{(1)}$ into the half-plane $S_k^{(j)}$ $(j = 1,2)$.

Suppose that a phase trajectory of the system (1.1), when $\mu = 0$, intersects the half-planes $S_k^{(j)}$ at the points $P_{k0}^{(j)}(x_k, y_{k0}^{(j)}, \tau_{k0}^{(j)})$, and when $\mu \neq 0$ at the points $P_k^{(j)}(x_k, y_k^{(j)}, \tau_k^{(j)})$. We shall prove that

$$\begin{aligned} y_k^{(j)} &= y_{k0}^{(j)} + \frac{\mu}{y_{k0}^{(j)}} \int\limits_{L_k^{(j)}} f(x, y, \tau + t_0)\, dx + \mu^2 (\ldots) \\ \tau_k^{(j)} &= \tau_{k0}^{(j)} + \mu \Phi_k^{(j)}(y_0, t_0) + \mu^2 (\ldots) \end{aligned} \tag{1.3}$$

Here $L_k^{(j)}$ is an integral curve of the system (1.1) when $\mu = 0$ which joins the point $P_0(x_0, y_0, 0)$ to the point $P_{k0}^{(j)}(x_k, y_{k0}^{(j)}, \tau_{k0}^{(j)})$.

We will prove first that the formulas (1.3) are valid if one transforms the half-plane $S_0^{(1)}$. For the sake of definiteness let us assume that in the strip between the half-planes $S_0^{(1)}$ and $S_1^{(1)}$ the function $\psi(x)$ has the form

$$\psi(x) \equiv \alpha_1 x + \beta_1 \equiv \omega_1^2 x - a_1$$

The solution of the system (1.1), with $\mu = 0$, which satisfies the conditions (1.2) has the form

$$\begin{aligned} x &= \left(x_0 - \frac{a_1}{\omega_1^2}\right)\cos \omega_1 \tau + \frac{y_0}{\omega_1} \sin \omega_1 \tau + \frac{a_1}{\omega_1^2} \equiv \alpha_{10}(\tau, y_0) \\ y &= -\left(x_0 - \frac{a_1}{\omega_1^2}\right)\omega_1 \sin \omega_1 \tau + y_0 \cos \omega_1 \tau \equiv \beta_{10}(\tau, y_0) \end{aligned}$$

Representing the solution of the system (1.1) with $\mu \neq 0$, which satisfies the condition (1.2) in the form of a power series in μ, we obtain

$$\begin{aligned} x &= \alpha_{10}(\tau, y_0) + \frac{\mu}{\omega_1}\int_0^\tau f_1^{(1)}[\alpha_{10}(u, y_0), \beta_{10}(u, y_0), u + t_0] \sin \omega_1 (\tau - u)\, du + \\ &\qquad + \mu^2 (\ldots) \\ y &= \beta_{10}(\tau, y_0) + \mu \int_0^\tau f_1^{(1)}[\alpha_{10}(u, y_0), \beta_{10}(u, y_0), u + t_0] \cos \omega_1 (\tau - u)\, du + \\ &\qquad + \mu^2 (\ldots) \end{aligned} \tag{1.4}$$

Let $\tau_1^{(1)}$ be the smallest time interval during which a mapped point reaches the half-plane $S_1^{(1)}$. We can express $\tau_1^{(1)}$ in the form of a series

$$\tau_1^{(1)} = \tau_{10}^{(1)} + \mu\Phi_1^{(1)}(y_0, t_0) + \ldots$$

Substituting $\tau = \tau_1^{(1)}$ into the equation (1.4) we obtain

$$y_1^{(1)} = y_{10}^{(1)} + \frac{\mu}{y_{10}^{(1)}} \int_0^{\tau_{10}^{(1)}} f_1^{(1)}[\alpha_{10}(u, y_0), \beta_{10}(u, y_0), u + t_0] \times$$
$$\times \left[\left(\omega_1 x_1 - \frac{a_1}{\omega_1}\right) \sin\ \omega_1(\tau_{10}^{(1)} - u) + y_{10}^{(1)} \cos \omega_1(\tau_{10}^{(1)} - u)\right] du + \mu^2(\ldots) =$$
$$= y_{10}^{(1)} + \frac{\mu}{y_{10}^{(1)}} \int_{L_1^{(1)}} f_1^{(1)}(x, y, \tau + t_0)\, dx + \mu^2(\ldots)$$
$$\tau_1^{(1)} = \tau_{10}^{(1)} - \frac{\mu}{y_{10}^{(1)}\omega_1} \int_0^{\tau_{10}^{(1)}} f_1^{(1)}[\alpha_{10}(u, y_0), \beta_{10}(u, y_0), u + t_0] \times$$
$$\times \sin \omega_1(\tau_{10}^{(1)} - u)\, du + \mu^2(\ldots) \equiv \tau_{10}^{(1)} + \mu\Phi_1^{(1)}(y_0, t_0) + \mu^2(\ldots)$$

Here $L_1^{(1)}$ is a space curve $x = \alpha_{10}(\tau, y_0)$, $y = \beta_{10}(\tau, y_0)$ which extends from the point $P_0(x_0, y_0, 0)$ to the point $P_{10}^{(1)}(x_1, y_{10}^{(1)}, \tau_{10}^{(1)})$. Analogously, one can prove the validity of the formulas (1.3) of the transformation of the half-plane $S_0^{(1)}$ into $S_1^{(1)}$ also for the case when

$$\psi(x) \equiv \alpha_1 x + \beta_1 \equiv -\omega_1^2 x + a_1$$

Let us suppose that the formulas (1.3) are valid when one transforms the half-plane $S_0^{(1)}$ into $S_{k-1}^{(1)}$. We will prove that they are also valid when one transforms the half-plane $S_0^{(1)}$ into $S_k^{(1)}$. For the sake of definiteness let

$$\psi(x) \equiv \alpha_k x + \beta_k \equiv -\omega_k^2 x + a_k, \quad f(x, y, \tau + t_0) \equiv f_k^{(1)}(x, y, \tau + t_0)$$
$$\text{if} \quad x_{k-1} < x < x_k$$

Expanding into a power series in μ the solution of the system (1.1), which satisfies the boundary conditions

$$x = x_{k-1}, \qquad y = y_{k-1}^{(1)} \qquad \text{if} \quad \tau = \tau_{k-1}^{(1)} \tag{1.5}$$

we obtain

$$x = \alpha_{k0}(\tau - \tau_{k-1}^{(1)}, y_{k-1}^{(1)}) +$$
$$+\frac{\mu}{\omega_k}\int_{\tau_{k-1}^{(1)}}^{\tau} f_k^{(1)}[\alpha_{k0}(u-\tau_{k-1}^{(1)}, y_{k-1}^{(1)}), \beta_{k0}(u-\tau_{k-1}^{(1)}, y_{k-1}^{(1)}), u+t_0]\sinh\omega_k(\tau-u)du+\mu^2(\ldots)$$
$$y = \beta_{k0}(\tau - \tau_{k-1}^{(1)}, y_{k-1}^{(1)}) + \tag{1.6}$$
$$+\mu\int_{\tau_{k-1}^{(1)}}^{\tau} f_k^{(1)}[\alpha_{k0}(u-\tau_{k-1}^{(1)}, y_{k-1}^{(1)}), \beta_{k0}(u-\tau_{k-1}^{(1)}, y_{k-1}^{(1)}), u+t_0]\cosh\omega_k(\tau-u)du+\mu^2(\ldots)$$

Here $x = \alpha_{k0}(\tau - \tau_{k-1}^{(1)}, y_{k-1}^{(1)})$, $y = \beta_{k0}(\tau - \tau_{k-1}^{(1)}, y_{k-1}^{(1)})$ is a solution of the system (1.1), with $\mu = 0$, satisfying conditions (1.5).

Let $\tau_k^{(1)}$ be the instant of time when the mapped point reaches the half-plane $S_k^{(1)}$. We express $\tau_k^{(1)}$ in the form

$$\tau_k^{(1)} = \tau_{k0}^{(1)} + \mu\Phi_k^{(1)}(y_0, t_0) + \ldots$$

Substituting $\tau = \tau_k^{(1)}$ into the equations (1.6), and expanding the right-hand sides into a power series in μ, we obtain

$$y_k^{(1)} = y_{k0}^{(1)} + \frac{\mu}{y_{k0}^{(1)}}\int_{L_{k-1}^{(1)}} f(x, y, \tau + t_0)\, dx +$$
$$+\frac{\mu}{y_{k0}^{(1)}}\int_{\tau_{k-1,0}^{(1)}}^{\tau_{k0}^{(1)}} f_k^{(1)}[\alpha_{k0}(u - \tau_{k-1,0}^{(1)}, y_{k-1,0}^{(1)}), \beta_{k0}(u - \tau_{k-1,0}^{(1)}, y_{k-1,0}^{(1)}), u + t_0]\times$$
$$\times [y_{k0}^{(1)}\cosh\omega_k(\tau_{k0}^{(1)} - u) - \omega_k(x_k - a_k/\omega_k^2)\sinh\omega_k(\tau_{k0}^{(1)} - u)]du + \mu^2(\ldots) =$$
$$= y_{k0}^{(1)} + \frac{\mu}{y_{k0}^{(1)}}\int_{L_k^{(1)}} f(x, y, \tau + t_0)\, dx + \mu^2(\ldots)$$

$$\tau_k^{(1)} = \tau_{k0}^{(1)} + \mu\Big\{\Phi_{k-1}^{(1)}(y_0, t_0) - \frac{1}{y_{k-1,0}^{(1)}y_{k0}^{(1)}}\frac{\partial\alpha_{k0}\left(\tau_{k0}^{(1)} - \tau_{k-1,0}^{(1)}, y_{k-1,0}^{(1)}\right)}{\partial y_{k-1,0}^{(1)}}\times$$
$$\times\int_{L_{k-1}^{(1)}} f(x, y, \tau + t_0)\, dx - \frac{1}{y_{k0}^{(1)}}\int_{\tau_{k-1,0}^{(1)}}^{\tau_{k0}^{(1)}}\frac{\partial\alpha_{k0}\left(\tau_{k0}^{(1)} - u, y_{k-1,0}^{(1)}\right)}{\partial y_{k-1,0}^{(1)}}\times$$
$$\times f_k^{(1)}[\alpha_{k0}(u - \tau_{k-1,0}^{(1)}, y_{k-1,0}^{(1)}), \beta_{k0}(u - \tau_{k-1,0}^{(1)}, y_{k-1,0}^{(1)}), u + t_0]\, du\Big\} +$$
$$+\mu^2(\ldots) \equiv \tau_{k0}^{(1)} + \mu\Phi_k^{(1)}(y_0, t_0) + \mu^2(\ldots)$$

In an analogous manner one can prove the correctness of the formulas (1.3) when

$$\psi(x) = \alpha_k x_k + \beta_k \equiv \omega_k^2 x - a_k$$

Suppose that in the continued motion the trajectory of the system (1.1) when $\mu \neq 1$ intersects the plane $y = 0$ at a point $M(x_k^*, 0, \tau_k^*)$, and when $\mu = 0$ at the point $M_0(x_{k0}^*, 0, \tau_{k0}^*)$ whereby $x_k \leqslant x_k^* \leqslant x_{k+1}$, $x_k \leqslant x_{k0}^* \leqslant x_{k+1}$.

For the sake of definiteness let us assume that

$$\psi(x) \equiv \alpha_{k+1} x + \beta_{k+1} \equiv -\omega_{k+1}^2 x + a_{k+1} \qquad \text{if} \quad x_k < x < x_{k+1}$$

The solution of the system (1.1), with $\mu \neq 0$, which satisfies the conditions

$$x = x_k, \qquad y = y_k^{(1)} \qquad \text{if} \quad \tau = \tau_k^{(1)} \tag{1.7}$$

has the form

$$x = \alpha_{k+1,0}(\tau - \tau_k^{(1)}, y_k^{(1)}) + \frac{\mu}{\omega_{k+1}} \int_{\tau_k^{(1)}}^{\tau} \sinh \omega_{k+1}(\tau - u) \times$$

$$\times f_{k+1}^{(1)}[\alpha_{k+1,0}(u - \tau_k^{(1)}, y_k^{(1)}), \beta_{k+1,0}(u - \tau_k^{(1)}, y_k^{(1)}), u + t_0]\, du + \mu^2(\ldots)$$

$$y = \beta_{k+1,0}(\tau - \tau_k^{(1)}, y_k^{(1)}) + \mu \int_{\tau_k^{(1)}}^{\tau} \cosh \omega_{k+1}(\tau - u) \times \tag{1.8}$$

$$\times f_{k+1}^{(1)}[\alpha_{k+1,0}(u - \tau_k^{(1)}, y_k^{(1)}), \beta_{k+1,0}(u - \tau_k^{(1)}, y_k^{(1)}), u + t_0]\, du + \mu^2(\ldots)$$

Here $\alpha_{k+1,0}(\tau - \tau_k^{(1)}, y_k^{(1)})$, $\beta_{k+1,0}(\tau - \tau_k^{(1)}, y_k^{(1)})$ is a solution of the system (1.1), with $\mu = 0$, satisfying conditions (1.7). Let us express τ_k^* in the forms $\tau_k^* = \tau_{k0}^* + \mu\Phi_k^*(y_0, t_0) + \ldots$ Substituting $\tau = \tau_k^*$ into the first one of the equations (1.8) and expanding the obtained expressions in powers of μ, we have

$$x_k^* = x_{k0}^* + \mu \left\{ \frac{\sinh \omega_{k+1}(\tau_{k0}^* - \tau_{k0}^{(1)})}{\omega_{k+1} y_{k0}^{(1)}} \int_{L_k^{(1)}} f(x, y, \tau + t_0)\, dx + \right.$$

$$+ \frac{1}{\omega_{k+1}} \int_{\tau_{k0}^{(1)}}^{\tau_{k0}^*} \sinh \omega_{k+1}(\tau_{k0}^* - u) \times$$

$$\times f_{k+1}^{(1)}\,[\alpha_{k+1,0}(u-\tau_{k0}^{(1)}, y_{k0}^{(1)}), \beta_{k+1,0}(u-\tau_{k0}^{(1)}, y_{k0}^{(1)}), u+t_0]du\Big\} + \mu^2(\ldots)$$

The solution of the system (1.1), satisfying the conditions

$$x = x_k^*, \quad y = 0 \quad \text{if} \quad \tau = \tau_k^* \tag{1.9}$$

when $x_k < x < x_{k+1}$, $y < 0$, can be expressed in the form

$$x = \alpha'_{k+1,0}(\tau-\tau_k^*\, x_k^*) + \frac{\mu}{\omega_{k+1}} \int_{\tau_k^*}^{\tau} \sinh\omega_{k+1}(\tau-u) \times \tag{1.10}$$

$$\times f_{k+1}^{(2)}\,[\alpha'_{k+1,0}(u-\tau_k^*, x_k^*),\ \beta'_{k+1,0}(u-\tau_k^*, x_k^*), u+t_0]\,du + \mu^2(\ldots)$$

$$y = \beta'_{k+1,0}(\tau-\tau_k^*, x_k^*) + \mu \int_{\tau_k^*}^{\tau} \cosh\omega_{k+1}(\tau-u) \times$$

$$\times f_{k+1}^{(2)}\,[\alpha'_{k+1,0}(u-\tau_k^*, x_k^*),\ \beta'_{k+1,0}(u-\tau_k^*, x_k^*), u+t_0]\ du + \mu^2(\ldots)$$

Here $\alpha'_{k+1,0}(\tau-\tau_k^*, x_k^*)$, $\beta'_{k+1,0}(\tau-\tau_k^*, x_k^*)$ is a solution of the system (1.1), with $\mu = 0$, which satisfies the initial condition (1.9).

Let $\tau_k^{(2)}$ be the instant of time when the mapped point reaches the half-plane $S_k^{(2)}$. We express $\tau_k^{(2)}$ in the form

$$\tau_k^{(2)} = \tau_{k0}^{(2)} + \mu\Phi_k^{(2)}(y_0, t_0) + \ldots$$

Substituting $\tau = \tau_k^{(2)}$ into the equation (1.10) and keeping in mind that $x(\tau_k^{(2)}) = x_k$, we obtain

$$y_k^{(2)} = y_{k0}^{(2)} + \frac{\mu}{y_{k0}^{(2)}} \int_{L_k^{(1)}} f(x, y, \tau+t_0)\,dx +$$

$$+\frac{\mu}{y_{k0}^{(2)}}\Big\{\int_{\tau_{k0}^{(1)}}^{\tau_{k0}^*} f_{k+1}^{(1)}\,[\alpha_{k+1,0}(u-\tau_{k0}^{(1)}, y_{k0}^{(1)}),\ \beta_{k+1,0}(u-\tau_{k0}^{(1)}, y_{k0}^{(1)}), u+t_0] \times$$

$$\times\ \omega_{k+1}\Big(x_{k0}^* - \frac{a_{k+1}}{\omega_{k+1}^2}\Big)\sinh\omega_{k+1}(u-\tau_{k0}^*)\,du +$$

$$+ \int_{\tau_{k0}^*}^{\tau_{k0}^{(2)}} f_{k+1}^{(2)}\,[\alpha_{k+1,0}(u-\tau_{k0}^*, x_{k0}^*), \beta_{k+1,0}(u-\tau_{k0}^*, x_{k0}^*), u+t_0] \times$$

$$\times \left[y_{k0}^{(2)} \cosh \omega_{k+1} (u - \tau_{k0}^{(2)}) + \omega_{k+1} \left(x_k - \frac{a_{k+1}}{\omega_{k+1}^2} \right) \sinh \omega_{k+1} (u - \tau_{k0}^{(2)}) \right] du \Big\} +$$

$$+ \mu^2 (\ldots) = y_{k0}^{(2)} + \frac{\mu}{y_{k0}^{(2)}} \int\limits_{L_k^{(2)}} f(x, y, \tau + t_0)\, dx + \mu^2 (\ldots)$$

$$\tau_k^{(2)} = \tau_{k0}^{(2)} + \mu \Big\{ \Phi_k^{(1)} (y_0, t_0) - \frac{1}{y_{k0}^{(2)} y_{k0}^{(1)}} \frac{\partial \alpha_{k+1,0} (\tau_{k0}^{(2)} - \tau_{k0}^{(1)}, y_{k0}^{(1)})}{\partial y_{k0}^{(1)}} \times$$

$$\times \int\limits_{L_k^{(1)}} f(x, y, \tau + t_0)\, dx - \frac{1}{y_{k0}^{(2)}} \int\limits_{\tau_{k0}^{(1)}}^{\tau_{k0}^{*}} \frac{\partial \alpha_{k+1,0} (\tau_{k0}^{(2)} - u, y_{k0}^{(1)})}{\partial y_{k0}^{(1)}} \times$$

$$\times f_{k+1}^{(1)} [\alpha_{k+1,0} (u - \tau_{k0}^{(1)}, y_{k0}^{(1)}), \ \beta_{k+1,0} (u - \tau_{k0}^{(1)}, y_{k0}^{(1)}), \ u + t_0]\, du -$$

$$- \frac{1}{y_{k0}^{(2)}} \int\limits_{\tau_{k0}^{*}}^{\tau_{k0}^{(2)}} f_{k+1}^{(2)} [\alpha_{k+1,0}' (u - \tau_{k0}^{*}, x_{k0}^{*}), \beta_{k+1,0}' (u - \tau_{k0}^{*}, x_{k0}^{*}), u + t_0] \times$$

$$\times \frac{\partial \alpha_{k+1,0} (\tau_{k0}^{(2)} - u, y_{k0}^{(1)})}{\partial y_{k0}^{(1)}}\, du \Big\} + \mu^2 (\ldots) \equiv \tau_{k0}^{(2)} + \mu \Phi_k^{(2)} (y_0, t_0) + \mu^2 (\ldots)$$

Hence, the formulas (1.3) retain their form also when the trajectory intersects the plane $y = 0$.

Returning to the formulas (1.3) at the earlier time t, we have

$$y_k^{(j)} = y_{k0}^{(j)} + \frac{\mu}{y_{k0}^{(j)}} \int\limits_{L_k^{(j)}} f(x, y, t)\, dx + \mu^2 (\ldots)$$

$$t_k^{(j)} \equiv t_0 + \tau_k^{(j)} = t_0 + \tau_{k0}^{(j)} + \mu \Phi_k^{(j)} (y_0, t_0) + \mu^2 (\ldots)$$

Let us now consider a mapping (transformation) of the half-plane $S_0^{(1)}$ into itself. We assume that when $\mu = 0$ the system (1.1) has a family of periodic solutions $L(y_0, t_0)$, which depends on the parameters y_0 and t_0 and which is such that $T'(y_0) \neq 0$. The point transformation of the half-plane $S_0^{(1)}$ into itself has, in the neighborhood of the curve L which passes through the points $P_0(x_0, y_0, t_0)$ and $P_0^{(1)}[x_0, y_0, t_0 + mT(y_0)]$, the following form

$$y_0^{(1)} = y_0 + \frac{\mu}{y_0} \int\limits_{L} f(x, y, t)\, dx + \mu^2 (\ldots) \equiv y_0 + \mu F (y_0, t_0) + \mu^2 (\ldots)$$

$$t_0^{(1)} = t_0 + mT (y_0) + \mu \Phi (y_0, t_0) + \mu^2 (\ldots) \tag{1.11}$$

where $T(y_0)$ is the period of the periodic solution of the system (1.1), with $\mu = 0$. This period depends on y_0. The number m is the number of turns of the curve L around the t-axis.

Obviously, the following theorem is true.

Theorem 1.1. In order that the point transformation (1.11) may have a fixed point

$$P_0\,(x_0, y_0^\circ + \mu y_1, t_0^\circ + \mu t_1)$$

which tends to the point $P(x_0, y_0^\circ, t_0^\circ)$ when μ goes to zero, it is necessary that the following conditions be fulfilled

$$F\,(y_0^\circ, t_0^\circ) \equiv \frac{1}{y_0^\circ}\int\limits_L f\,(x, y, t)\,dx = 0, \qquad T\,(y_0^\circ) = \frac{2\pi n}{m} \tag{1.12}$$

where L is a closed integral curve of the system (1.1) when $\mu = 0$, which passes through the points $P(x_0, y_0^\circ, t_0^\circ)$, $P^{(1)}(x_0, y_0^\circ, t_0^\circ + 2\pi n)$, and where n/m is a rational fraction.

Theorem 1.2. Let y_0° and t_0° be a solution of the system (1.12). If

$$T'\,(y_0^\circ)\,F_{t_0}'\,(y_0^\circ, t_0^\circ) \neq 0$$

then the transformation (1.11) has a single fixed point

$$P_0\,(x_0, y_0^\circ + \mu y_1, t_0^\circ + \mu t_1)$$

which tends to the point $P(x_0, y_0^\circ, t_0^\circ)$ when μ goes to zero.

Let us introduce the notation

$$\alpha\,(y_0, t_0) \equiv \mu F\,(y_0, t_0) + \mu^2\,(\ldots)$$
$$\beta\,(y_0, t_0) \equiv -\,2\pi n + T\,(y_0)\,m + \mu\Phi\,(y_0, t_0) + \mu^2\,(\ldots)$$

The Jacobian

$$\frac{\partial\,(\alpha, \beta)}{\partial\,(y_0, t_0)} = -\,\mu m T'\,(y_0^\circ)\,F_{t_0}'\,(y_0^\circ, t_0^\circ) + \mu^2\,(\ldots)$$

evaluated at the point $y_0 = y_0^\circ + \mu y_1$, $t_0 = t_0^\circ + \mu t_1$ is different from zero if μ is sufficiently small. Hence, for small values of μ, the system

$$y_0 = y_0 + \mu F\,(y_0, t_0) + \mu^2\,(\ldots)$$
$$t_0 + 2\pi n = t_0 + mT\,(y_0) + \mu\Phi\,(y_0, t_0) + \mu^2\,(\ldots)$$

is solvable for y_0 and t_0, and the transformation (1.11) has a fixed point $P_0(x_0, y_0, t_0)$.

We shall elucidate further the conditions of stability of the fixed point for the transformation (1.11). The characteristic equation of the point transformation (1.11) is given (with an accuracy of second order infinitesimals in μ) by

$$\lambda^2 + p\lambda + q = 0$$

where

$$p = -2 - \mu\,[\Phi_{t_0}'(y_0^\circ, t_0^\circ) + F_{y_0}'(y_0^\circ, t_0^\circ)] + \mu^2(\ldots)$$
$$q = 1 + \mu\,[-mT'(y_0^\circ)\,F_{t_0}'(y_0^\circ, t_0^\circ) + \Phi_{t_0}'(y_0^\circ, t_0^\circ) + F_{y_0}'(y_0^\circ, t_0^\circ)] + \mu^2(\ldots)$$

The conditions for stability of the fixed point of the transformation (1.11) are

$$1 + p + q \equiv -\mu m T'(y_0^\circ)\,F_{t_0}'(y_0^\circ, t_0^\circ) + \mu^2(\ldots) > 0$$
$$1 - p + q \equiv 4 + \mu(\ldots) > 0$$
$$q - 1 \equiv \mu\,[-mT'(y_0^\circ)\,F_{t_0}'(y_0^\circ, t_0^\circ) + \Phi_{t_0}'(y_0^\circ, t_0^\circ) + F_{y_0}'(y_0^\circ, t_0^\circ)] + \mu^2(\ldots) < 0$$

These conditions will be fulfilled for small enough μ if

$$T'(y_0^\circ)\,F_{t_0}'(y_0^\circ, t_0^\circ) < 0$$
$$-mT'(y_0^\circ)\,F_{t_0}'(y_0^\circ, t_0^\circ) + \Phi_{t_0}'(y_0^\circ, t_0^\circ) + F_{y_0}'(y_0^\circ, t_0^\circ) < 0$$

The fixed point will be unstable if one of these inequalities is violated.

Let us assume further that when $\mu = 0$ the system (1.1) has a family of periodic solutions $L(y_0, t_0)$ whose period depends on y_0.

In this case the curves L must lie entirely between two planes $x = x_{-1} < 0$ and $x = x_1 > 0$, where the function $\psi(x)$ has the form

$$\psi(x) = \begin{cases} \omega_1^2 x & \text{if } x_{-1} < x < 0 \\ \omega_2^2 x & \text{if } 0 < x < x_1 \end{cases}$$

In this case the point transformation of the half-plane $S_0^{(1)}(x = 0, y > 0)$ into itself in the neighborhood of the curve L which passes through the points $P_0(0, y_0, t_0)$ and $P_0^{(1)}[0, y_0, t_0 + m(\pi/\omega_1 + \pi/\omega_2)]$, has the form

$$y_0^{(1)} = y_0 + \frac{\mu}{y_0}\int_L f(x, y, t)\,dx + \mu^2(\ldots) \equiv y_0 + \mu F(y_0, t_0) + \mu^2(\ldots)$$

$$t_0^{(1)} = t_0 + m\left(\frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}\right) + \tag{1.13}$$

$$+\frac{\mu}{y_0}\left\{\frac{1}{\omega_2}\sum_{i=0}^{m-1}(-1)^i\int_{L_{i1}} f(x, y, t)\sin\omega_2\left(t - i\frac{\pi}{\omega_1} - t_0\right)dt +\right.$$

$$\left.+\frac{1}{\omega_1}\sum_{i=0}^{m-1}(-1)^{i+1}\int_{L_{i2}} f(x, y, t)\sin\omega_1\left[t - (i+1)\frac{\pi}{\omega_2} - t_0\right]dt\right\} + \mu^2(\ldots) \equiv$$

$$\equiv t_0 + m\left(\frac{\pi}{\omega_1} + \frac{\pi}{\omega_2}\right) + \mu\Phi(y_0, t_0) + \mu^2(\ldots)$$

Here m is the number of turns of the integral curve L around the t-axis; L_{i1} is the piece of the integral curve L included between the planes

$$t = t_0 + (\pi/\omega_1 + \pi/\omega_2)\,i + \pi/\omega_2, \qquad t = t_0 + (\pi/\omega_1 + \pi/\omega_2)\,i$$

while L_{i2} is the piece of the integral curve L included between the planes

$$t = t_0 + (\pi/\omega_1 + \pi/\omega_2)\,i + \pi/\omega_2, \qquad t = t_0 + (\pi/\omega_1 + \pi/\omega_2)(i+1)$$

Obviously, if $\omega_1 = \omega_2$ then

$$\Phi(y_0, t_0) = \frac{1}{y_0\omega_1}\int_L f(x, y, t)\sin\omega_1(t - t_0)\,dt$$

It follows from the expression (1.13) that the point transformation (1.13) can possess a fixed point only if ω_1 and ω_2 are rational numbers.

Let us assume that the number m satisfies the condition

$$m\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) = 2n$$

where n is a positive integer.

Then we can obtain the next theorem in an entirely analogous manner.

Theorem 1.3. In order that the point transformation (1.13) may have a fixed point $P_0(0,\ y_0^{\circ} + \mu y_1,\ t_0^{\circ} + \mu t_1)$ which tends to the point $P(0,\ y_0^{\circ},\ t_0^{\circ})$ when μ approaches zero, it is necessary that

$$F(y_0^\circ, t_0^\circ) = 0, \qquad \Phi(y_0^\circ, t_0^\circ) = 0 \tag{1.14}$$

Theorem 1.4. Let y_0° and t_0° be a solution of the system (1.14). If

$$F_{t_0}'(y_0^\circ, t_0^\circ)\,\Phi_{y_0}'(y_0^\circ, t_0^\circ) - F_{y_0}'(y_0^\circ, t_0^\circ)\,\Phi_{t_0}'(y_0^\circ, t_0^\circ) \neq 0$$

then the transformation (1.13) has a single fixed point

$$P_0(0, y_0^\circ + \mu y_1, t_0^\circ + \mu t_1)$$

which tends to the point $P(0, y_0^\circ, t_0^\circ)$ when μ approaches zero. This fixed point is stable if

$$F_{y_0}'(y_0^\circ, t_0^\circ)\,\Phi_{t_0}'(y_0^\circ, t_0^\circ) - F_{t_0}'(y_0^\circ, t_0^\circ)\,\Phi_{y_0}'(y_0^\circ, t_0^\circ) > 0$$
$$\Phi_{t_0}'(y_0^\circ, t_0^\circ) + F_{y_0}'(y_0^\circ, t_0^\circ) < 0$$

and it is unstable if any one of these inequalities is violated.

2. Autonomous system which is near a piece-wise linear one. We shall now consider the system

$$dx/dt = y, \qquad dy/dt = -\psi(x) + \mu f(x, y) \tag{2.1}$$

Let

$$\begin{aligned} &\psi(x) \equiv \alpha_i x + \beta_i && \text{if } x_{i-1} < x < x_i \\ &f(x, y) = f_i^{(1)}(x, y) && \text{if } x_{i-1} < x < x_i,\ y > 0 \qquad (i = \ldots -1, 0, 1 \ldots) \\ &f(x, y) = f_i^{(2)}(x, y) && \text{if } x_{i-1} < x < x_i,\ y < 0 \end{aligned}$$

The functions $f_i^{(j)}(x, y)$ $(j = 1, 2)$ are analytic in x and y, while μ is a small positive parameter.

Let us denote by $S_i^{(1)}$ the half-lines $x = x_i$ when $y > 0$, and by $S_i^{(2)}$ the half-lines $x = x_i$ when $y < 0$, and let us consider the phase trajectories of the system (2.1) when $\mu = 0$ and when $\mu \neq 0$ which satisfy the initial conditions

$$x = x_0, \quad y = y_0 \quad \text{if } t = 0 \tag{2.2}$$

Suppose that the phase trajectory of the system (2.1) with $\mu = 0$ intersects the half-lines $S_k^{(j)}$ at the points $P_{k0}^{(j)}(x_k, y_{k0}^{(j)})$, and when $\mu \neq 0$ it intersects them at the points $P_k^{(j)}(x_k, y_k^{(j)})$.

Let us assume that when $\mu = 0$ the system (2.1) has a family of periodic solutions $L(y_0)$ which depend on the parameter y_0. Then the point transformation of the half-line $S_0^{(1)}$ into itself in the neighborhood of the closed curve L will have the form

$$y_0^{(1)} = y_0 + \frac{\mu}{y_0}\int_L f(x, y)\, dx + \mu^2 (\ldots) \equiv y_0 + \mu F(y_0) + \mu^2 (\ldots) \qquad (2.3)$$

where $L = L(y_0)$ is a closed integral curve passing through the point $P_0(x_0, y_0)$.

The following theorems are valid.

Theorem 2.1. In order that the transformation (2.3) with μ small enough may have a fixed point

$$P_0(x_0, y_0^\circ + \mu y_1)$$

which tends to $P(x_0, y_0^\circ)$ when μ goes to zero, it is necessary that the condition

$$F(y_0^\circ) = 0 \qquad (2.4)$$

be satisfied.

Theorem 2.2. Let y_0° be a solution of the equation (2.4). If $F'(y_0^\circ) \neq 0$ then the transformation (2.3) has a single fixed point $P_0(x_0, y_0^\circ + \mu y_1)$ which tends to the point $P(x_0, y_0^\circ)$ when μ goes to zero. This fixed point is stable if $F'(y_0^\circ) < 0$, and it is unstable if $F'(y_0^\circ) > 0$.

The obtained conditions for the existence and stability of a periodic solution of the system (2.1) are analogous to the corresponding conditions given in [4] for systems which are near to Hamiltonian systems.

If the functions $\psi(x)$ and $f(x, y)$ are of period 2π in x, then the phase space of the system (2.1) will be cylindrical if one considers the lines $x = x_0$ and $x = 2\pi + x_0$ as coincident. Theorems 2.1 and 2.2 in this case will yield necessary and sufficient conditions for the existence of a fixed point which corresponds to the periodic solution that encloses the cylinder. The curve $L(y_0^\circ)$ in equation (2.4) in this case will be a closed integral curve of the system (2.1) with $\mu = 0$ which passes through the point $P(x_0, y_0^\circ)$ and goes around the cylinder.

3. *Example*. Let us consider an equation from the theory of electrical machines [5-8]

$$\ddot{\varphi} + \lambda[1 - \beta\Theta'(\varphi)]\dot{\varphi} + \Theta(\varphi) = \gamma \qquad (\lambda > 0,\ \gamma > 0)$$

where the function $\Theta(\varphi)$ is of period 2π, with the piece-wise linear approximation

$$\Theta(\varphi) = \Theta_1(\varphi) \equiv (-1)^k \frac{2}{\pi}\varphi + (-1)^{k-1} 2k$$

$$(2k-1)\frac{\pi}{2} < \varphi < (2k+1)\frac{\pi}{2} \qquad (k = \ldots -1, 0, 1 \ldots)$$

For the phase space we will take a strip located between the straight lines $\varphi = -\pi$ and $\varphi = \pi$. The points of these two lines with the same ordinates will be considered to be identical. We introduce a small positive parameter by setting $\lambda = \mu\lambda_0$, $\gamma = \mu\gamma_0$, and we go over to the system which is close to the piece-wise linear one

$$d\varphi / dt = y, \qquad dy / dt = -\Theta_1(\varphi) + \mu\{\gamma_0 - \lambda_0[1 - \beta\Theta_1'(\varphi)]\, y\} \tag{3.1}$$

A study of the periodic solutions of the system (3.1) makes it possible for us to form rigorously a qualitative picture of the division of the phase space into trajectories for small values of λ and γ, and to explain how this picture changes with a change of the parameters.

When $\mu = 0$ the trajectories of the system (3.1) have either the form of closed curves enclosing a state of equilibrium (of the center type) at the point $\varphi = 0$, $y = 0$, or they are closed curves made up of pieces of ellipses and hyperbolas enclosing the phase space (the cylinder). These two regions are separated by separatrices which are composed of pieces of straight lines and ellipses passing from saddle to saddle (in the points $(-\pi, 0)$, $(\pi, 0)$ the system (3.1) with $\mu = 0$ has simple saddles).

If $\mu \neq 0$, but arbitrarily small, the closed curves enclosing the state of equilibrium or the phase cylinder become spirals, and only certain ones of the integral curves remain closed, that is, they become limit cycles. The separatrices which together with the state of equilibrium form a closed contour when $\mu = 0$, will not form such a contour when $\mu \neq 0$; they, too, will become spirals that close in on a limit cycle or on a state of equilibrium, or else recede to infinity. A knowledge of the character and distribution of the limit cycles makes it possible to determine completely the qualitative structure of the division of the phase space into the trajectories.

The system (3.1) may have cycles that enclose the cylinder as well as cycles that enclose a state of equilibrium $O_1(\mu\gamma_0\pi/2, 0)$. We shall try to find the cycles that enclose the cylinder. If we apply Theorem 2,1, we obtain

$$F_1(y_0^\circ) = \frac{1}{y_0^\circ}\left\{2\int\limits_{L_1}\left[\gamma_0 - \lambda_0\left(1 + \beta\frac{2}{\pi}\right)y\right]d\varphi + 2\int\limits_{L_2}\left[\gamma_0 - \lambda_0\left(1 - \frac{2}{\pi}\beta\right)y\right]d\varphi\right\} \tag{3.2}$$

Here L_1 and L_2 are parts of the integral curve of the system (3.1) with $\mu = 0$ which passes through the point $P(-\pi,\ y_0^{\circ})$. These parts L_1 and L_2 are located in the intervals $-\pi \leqslant \varphi \leqslant -\pi/2$ and $-\pi/2 \leqslant \varphi \leqslant 0$, respectively. The equations of the curves L_1 and L_2 are

$$(L_1)\ \frac{y^2}{2} - \frac{(\varphi+\pi)^2}{\pi} = \frac{(y_0^{\circ})^2}{2}, \qquad (L_2)\ \frac{y^2}{2} + \frac{\varphi^2}{\pi} = \frac{(y_0^{\circ})^2}{2} + \frac{\pi}{2}$$

respectively.

The integration is performed in the direction of the motion along the trajectories. If $y_0^{\circ} > 0$, the evaluation of the integrals on the right-hand side of equation (3.2) yields

$$F_1(y_0^{\circ}) = \frac{1}{y_0^{\circ}}\left\{2\pi\gamma_0 - \lambda_0\sqrt{\frac{\pi}{2}}\left[2\sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}+2h} + \right.\right.$$
$$+ h\left(1+\frac{2}{\pi}\beta\right)\ln\left(\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\pi}{2}+2h}\right)^2\frac{1}{|2h|} +$$
$$\left.+\left(1-\beta\frac{2}{\pi}\right)(2h+\pi)\ \sin^{-1}\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{2h+\pi}}\right\} \equiv \frac{1}{y_0^{\circ}}\psi_1(h) \qquad \left(h=\frac{(y_0^{\circ})^2}{2}\right)$$

For the purpose of determining the number of roots of the equation

$$\psi_1(h) = 0 \tag{3.3}$$

we find

$$\psi_1'(h) = -\lambda_0\sqrt{\frac{\pi}{2}}\left[\left(1+\frac{2}{\pi}\beta\right)\ln\left(\sqrt{\frac{\pi}{2}}+\sqrt{\frac{\pi}{2}+2h}\right)^2\frac{1}{|2h|} + \right.$$
$$\left.+2\left(1-\frac{2}{\pi}\beta\right)\ \sin^{-1}\frac{\sqrt{\pi}}{\sqrt{2}\sqrt{2h+\pi}}\right], \qquad \psi_1''(h) = \frac{\sqrt{2}\,\pi\lambda_0}{\sqrt{\pi+4h}}\,\frac{4h+\pi+2\beta}{2h(2h+\pi)}$$

$$\lim_{h\to 0}\varphi_1(h) = 2\pi\gamma_0 + \lambda_0\sqrt{\frac{\pi}{2}}\,\frac{\pi}{2}\left(\beta-2-\frac{\pi}{2}\right), \qquad \lim_{h\to\infty}\psi_1(h) = -\infty$$

Having investigated the behavior of the function $\psi_1(h)$ in the interval $0 < h < \infty$, we come to the conclusion that for $0 < h < \infty$ the function $\psi_1(h)$ is monotonically decreasing if $\beta \geqslant -\pi/2$, and that it has one maximum if $\beta < -\pi/2$. This shows that if

$$4\gamma_0 + \lambda_0\sqrt{\pi/2}\,(\beta - 2 - \pi/2) > 0$$

the equation (3.3) has one positive root. The system (3.1) will then have one stable limit cycle which enclosed the cylinder in the upper half of the phase space. In the region of the space of the parameters λ_0, γ_0 and β, determined by the relations

$$\psi_1'(h) = 0, \qquad \psi_1(h) > 0$$
$$4\gamma_0 + \lambda_0 \sqrt{\frac{\pi}{2}} \left(\beta - 2 - \frac{\pi}{2}\right) < 0$$

the equation (3.3) has two positive roots. The system (3.1) then will have two limit cycles in the upper half of the phase space. Whereby to the larger root of equation (3.3) there will correspond a stable limit cycle, while to the smaller root, an unstable limit cycle.

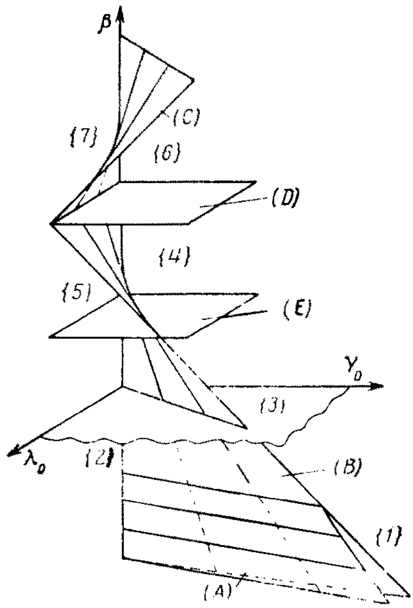

Fig. 1.

Let us now try to find the limit cycles which enclose the cylinder and are located in the lower half of the phase space where $y_0^\circ < 0$.

Integrating expression (3.2), and setting $(y_0^\circ)^2/2 = h$, we obtain

$$F_2(y_0^\circ) = \frac{1}{y_0^\circ}\left\{-2\pi\gamma_0 - \lambda_0 \frac{\sqrt{\pi}}{\sqrt{2}}\left[2\frac{\sqrt{\pi}}{\sqrt{2}}\sqrt{\frac{\pi}{2}+2h} + h\left(1+\frac{2}{\pi}\beta\right) \times \right.\right.$$
$$\left.\left. \times \ln\left(\frac{\sqrt{\pi}}{\sqrt{2}} + \sqrt{\frac{\pi}{2}+2h}\right)^2 \frac{1}{|2h|} + \left(1-\frac{2}{\pi}\beta\right)(2h+\pi) \quad \sin^{-1} \quad \frac{\sqrt{\pi}}{\sqrt{2}\sqrt{2h+\pi}}\right]\right\} \equiv \frac{\psi_2(h)}{y_0^\circ}$$

In a way analogous to the one above, one can show that for $\gamma_0 > 0$, $\lambda_0 > 0$ the system (3.1) will not have more than one limit cycle which encloses the cylinder in the lower half of the phase space. If

$$\lim_{h\to 0}\psi_2(h) = -2\pi\gamma_0 + \lambda_0\frac{\pi}{2}\sqrt{\frac{\pi}{2}}\left(\beta - 2 - \frac{\pi}{2}\right) > 0$$

then the system (3.1) will have a limit cycle enclosing the phase cylinder in the lower half of the phase space.

Finally, let us find a limit cycle that encloses a state of equilibrium.

Applying Theorem 2.1, we obtain

$$F_3(y_1^\circ) = -\frac{4\lambda_0}{y_1^\circ}\left[\left(1+\frac{2}{\pi}\beta\right)\int\limits_{L_1} y\,d\varphi + \left(1-\frac{2}{\pi}\beta\right)\int\limits_{L_2} y\,d\varphi\right] \equiv \frac{1}{y_1^\circ}\psi_3(h_1) \tag{3.4}$$

Here L_1 and L_2 are parts of an integral curve of the system (3.1) with $\mu = 0$ which passes through the points $P'(-\pi/2,\ y_1^{\circ})$ $(0 < y_1^{\circ} < \sqrt{(\pi/2)})$. These parts L_1 and L_2 are located in the intervals $-\pi \leqslant \varphi \leqslant -\pi/2$, and $-\pi/2 \leqslant \varphi \leqslant 0$, respectively.

The equations of L_1 and L_2 have the form

$$(L_1)\ \frac{y^2}{2} - \frac{(\varphi+\pi)^2}{\pi} = h_1, \qquad (L_2)\ \frac{y^2}{2} + \frac{\varphi^2}{\pi} = h_1 + \frac{\pi}{2}\left(-\frac{\pi}{4} \leqslant h_1 = \frac{(y_1^\circ)^2}{2} - \frac{\pi}{4} \leqslant 0\right) \tag{3.5}$$

From the expression (3.4) it follows that if $(1 - 2\beta/\pi)(1 + 2\beta/\pi) > 0$ then the equation $F_3(y_1^{\circ}) = 0$ has no real roots.

Suppose that $1 - 2\beta/\pi < 0$. It is easily seen that

$$\psi_3(h_1) = 2\psi_1(h_1) - 4\pi\gamma_0, \qquad \psi_3'(h_1) = 2\psi_1'(h_1), \qquad \psi_3''(h_1) = 2\psi_1''(h_1)$$

For values of h_1 which satisfy the condition (3.5), we have

$$\psi_3''(h_1) < 0, \qquad \psi_3'\left(-\frac{\pi}{4}\right) > 0, \qquad \lim_{h_1\to 0}\psi_3'(h_1) = -\infty$$

$$\psi_3\left(-\frac{\pi}{4}\right) = -2\lambda_0\sqrt{\frac{\pi}{2}}\left(1-\frac{2}{\pi}\beta\right)\frac{\pi^2}{4} > 0, \qquad \lim_{h_1\to 0}\psi_3(h_1) = \lambda_0\sqrt{\frac{\pi}{2}}\,\pi\left(\beta - 2 - \frac{\pi}{2}\right)$$

From this we conclude that if $1 - 2\beta/\pi < 0$, and $\beta - 2 - \pi/2 < 0$, then there exists just one stable limit cycle which encloses a state of equilibrium. In an analogous manner it can be shown that if $1 - 2\beta/\pi > 0$ the system (3.1) has no cycles which enclose a state of equilibrium.

For the purpose of explaining the picture of the phase trajectories we note that the state of equilibrium $O_1(\mu\gamma_0\pi/2,\ 0)$ will be a stable focus if

$$\mu\lambda_0(1 - 2\beta/\pi) > 0$$

and it will be an unstable focus if $\mu\lambda_0(1 - 2\beta/\pi) < 0$.

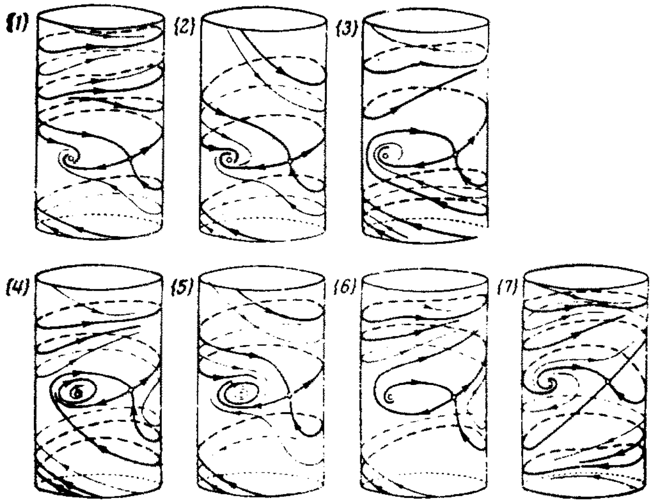

Fig. 2.

Figure 1 shows the division of the space of the parameters λ_0, γ_0 and β into regions that correspond to definite qualitative pictures of the phase trajectories.

In the region $\{1\}$

$$(4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) < 0,\ 1 + 2\beta/\pi < 0,\ \psi_1'(h) = 0,\ \psi_1(h) > 0)$$

the system (3.1) has two limit cycles that enclose the cylinder in the upper half of the phase space. The upper cycle is stable while the lower one is unstable.

In the region $\{2\}$

$$(\gamma_0 > 0,\ 4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) < 0,\ 1 - 2\beta/\pi > 0,\ \psi_1'(h) = 0,\ \psi_1(h) < 0)$$

the system (3.1) has no limit cycles.

In the region $\{3\}$

$$(\lambda_0 > 0,\ 4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) > 0,\ 1 - 2\beta/\pi > 0)$$

the system (3.1) has one stable cycle enclosing the cylinder in the upper half of the phase space.

In the region {4}

$$(\lambda_0 > 0,\ 4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) > 0,\ 1 - 2\beta/\pi < 0,\ \beta - 2 - \pi/2 < 0)$$

the system (3.1) has one stable limit cycle which encloses a state of equilibrium, and a stable limit cycle enclosing the cylinder in the upper half of the phase space.

In the region {5}

$$(\gamma_0 > 0,\ \lambda_0 > 0,\ 4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) < 0,\ 1 - 2\beta/\pi < 0)$$

the system (3.1) has one stable limit cycle enclosing a state of equilibrium.

In the region {6}

$$(\beta - 2 - \pi/2 > 0,\ \lambda_0 > 0,\ \gamma_0 > 0,\ -4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) < 0)$$

the system has one stable limit cycle in the upper half of the phase space.

In the region {7}

$$(\gamma_0 > 0,\ \lambda_0 > 0,\ -4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) > 0)$$

the system (3.1) has two stable cycles enclosing the cylinder. One of them is located in the lower, the other one in the upper half of the phase space.

The qualitative pictures of the phase trajectories for the enumerated regions are shown in Fig. 2.

Let us consider the qualitative pictures of the phase trajectories on the bifurcated surfaces.

On the surface (A), determined by the relations

$$\psi_1(h) = 0, \qquad \psi_1'(h) = 0$$

the system has a semistable limit cycle which encloses the cylinder in the upper half of the phase space.

On the surfaces (B) and (C)

$$4\gamma_0 + \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) = 0$$
$$4\gamma_0 - \lambda_0 \sqrt{\pi/2}\,(\beta - 2 - \pi/2) = 0$$

the limit cycles that enclose the cylinder in the upper or lower halves of the phase space will run into separatrices which go from saddle to saddle.

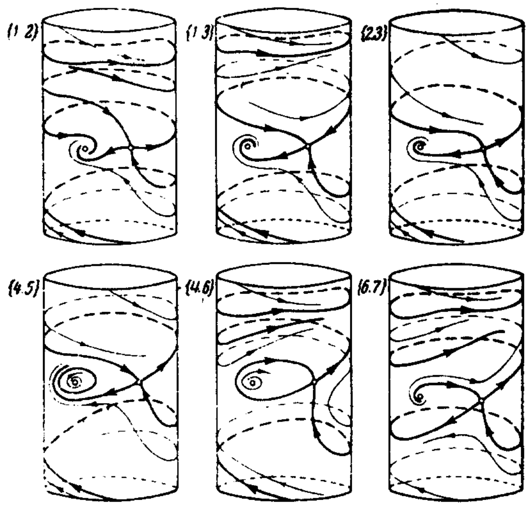

Fig. 3.

On the plane (D) $(\beta - 2 - \pi/2 = 0)$, the limit cycle that encloses the state of equilibrium $O_1(\mu\gamma_0\pi/2,\ 0)$ runs into the separatrix which passes from one saddle into the same saddle. On the surface (E) $(1 - 2\beta/\pi = 0)$ the system has a state of equilibrium of the center type at the point $O_1(\mu\gamma_0\pi/2,\ 0)$.

Figure 3 represents the qualitative pictures of the phase trajectories that correspond to the bifurcated values of the parameters. The numbers in the braces denote the regions on the common boundary of which the system (3.1) has the indicated qualitative picture for the phase trajectories.

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